

Vortex ratchet

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Abstract

We present a new class of thermal ratchets operating under the action of a symmetry breaking non-Hermitian perturbation which rectifies thermal fluctuations, and driven by a unbiased periodic force. The peculiar non-Hermitian dynamics which follows causes energy transduction from the force to the system in such a way that an average ‘uphill’ particle current is induced. We discuss physical realizations in assemblies of orientable particles, in itinerant oscillator models, and in problems of diffusion in disordered media

Key words: Brownian motor, transport processes, dissipation

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1 Introduction

In the last few years it has been shown the existence a wide variety of transport processes at the mesoscopic level in which thermal noise plays a decisive role. To understand how those processes work, several physical models and technological implementations have been proposed [1], [2]. The peculiar effect of thermal noise can be illustrated in thermal ratchets or Brownian motors, which have the ability of extracting work from out-of-equilibrium fluctuations in spatially periodic systems without spatial inversion symmetry. One way to do this is by the combination of a ratchet-like potential which rectifies thermal fluctuations, and a periodic unbiased force driving the system out of equilibrium.

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Our purpose in this paper is to propose a new class of Brownian motors able to extract work from thermal fluctuations in nonequilibrium systems driven by a periodic force. Unlike the ones previously introduced, the symmetry breaking is due to the presence of a vortex field responsible for the existence of a non-Hermitian component in the stochastic dynamics. Fluctuations are then rectified by a non-equilibrium source instead of a ratchet-like potential. We will call those devices *vortex ratchets*.

The paper is organized as follows. In section 2, we discuss the stochastic dynamics of these systems. We formulate the Fokker-Planck equation and compute the susceptibility. In section 3, we analyze the dissipation of energy in the system and introduce the ratchet current. Section 4 is devoted to discuss some applications

2 Stochastic dynamics

We consider a Brownian degree of freedom, parameterized by a coordinate \mathbf{x} , interacting with a thermal bath which is maintained out of equilibrium by the persisting action of an external drift $\mathbf{v}(\mathbf{x})$, [3]. This drift could represent, for example, a constant or a quenched velocity field or an external field. The stochastic dynamics is governed by the probability density $\Psi(\mathbf{x}, t)$ satisfying the conservation law

$$\partial_t \Psi(\mathbf{x}, t) + \nabla_{\mathbf{x}} \cdot (\mathbf{v}(\mathbf{x}) \Psi(\mathbf{x}, t)) = -\nabla_{\mathbf{x}} \cdot \mathbf{J}_{\psi}(\mathbf{x}, t) , \quad (1)$$

in which the probability current \mathbf{J}_{ψ} is given by

$$\mathbf{J}_{\psi}(\mathbf{x}, t) = -D \nabla_{\mathbf{x}} \Psi(\mathbf{x}, t) + b \mathbf{F}(\mathbf{x}, t) \Psi(\mathbf{x}, t) . \quad (2)$$

The dynamics of the Brownian degree of freedom is then governed by the Fokker-Planck equation

$$\partial_t \Psi(\mathbf{x}, t) = -\nabla_{\mathbf{x}} \cdot (\mathbf{v}(\mathbf{x}) \Psi - D \nabla_{\mathbf{x}} \Psi) - \nabla_{\mathbf{x}} \cdot b \mathbf{F}(\mathbf{x}, t) \Psi(\mathbf{x}, t) , \quad (3)$$

where b is the mobility, $D = k_B T b$ the corresponding diffusion coefficient, and $\mathbf{F}(\mathbf{x}, t) = \mathbf{F}_o(\mathbf{x}) \lambda(t)$ a periodic force, with $\lambda(t) = \lambda_o e^{i\omega t}$.

In the linear response regime the formal solution of the Fokker-Planck equation

reads

$$\Psi(\mathbf{x}, t) = \Psi_o(\mathbf{x}) + \int_{t_o}^t \lambda(t') e^{(t-t')\mathcal{L}_o} \mathcal{L}_1 \Psi_o(\mathbf{x}) dt' = \Psi_o(\mathbf{x}) + \Delta\Psi(\mathbf{x}, t), \quad (4)$$

where $\mathcal{L}_o = -\nabla_{\mathbf{x}} \cdot \mathbf{v} + D\nabla_{\mathbf{x}}^2$ is the unperturbed Fokker-Planck operator, and $\mathcal{L}_1 = -b\nabla_{\mathbf{x}} \cdot \mathbf{F}_o$ the perturbation. Moreover, $\Psi_o(\mathbf{x})$ corresponds to the stationary solution of eq. (3), [4].

The presence of the perturbation causes deviation in the coordinate, $\Delta\mathbf{x}$. To compute that quantity, we will expand the term $\mathcal{L}_1 \Psi_o(\mathbf{x})$ in series of the eigenfunctions of the operator \mathcal{L}_o , ϕ_n with eigenvalues a_n ; $n = 0, 1, \dots$

$$\mathcal{L}_1 \Psi_o(\mathbf{x}) = \sum_{n=0}^{\infty} \{c_n \phi_n(\mathbf{x}) + c_n^* \phi_n^*(\mathbf{x})\}, \quad (5)$$

where c_n are the corresponding coefficients. We obtain

$$\Delta\mathbf{x}(t) = \int \mathbf{x} \Delta\Psi(\mathbf{x}, t) d\mathbf{x} = \int_{t_o}^t d\tau \chi(t - \tau) \lambda(\tau), \quad (6)$$

which defines the susceptibility $\chi(t)$.

We will assume the existence of a dominant time scale governing the relaxation process, corresponding to the $n = 1$ mode in the expansion (5). Since the remaining modes decay faster, we can truncate the series retaining only the second term. Thus, considering only contributions of the first mode the susceptibility is given by $\chi(t) = \mathbf{A}e^{a_1 t} + c.c.$, with \mathbf{A} defined as

$$\mathbf{A} = c_1 \int \mathbf{x} \phi_1(\mathbf{x}) d\mathbf{x} \quad (7)$$

Assuming now $t_o \rightarrow -\infty$, eq. (6) can be rewritten as

$$\Delta\mathbf{x}(t) = \chi(\omega) \lambda(t), \quad (8)$$

where $\chi(\omega)$ is the Fourier transform of $\chi(t)$ given by

$$\chi(\omega) = \frac{\mathbf{A}}{I_1} \frac{1}{\beta - i(\alpha + 1)} + \frac{\mathbf{A}^*}{I_1} \frac{1}{\beta - i(\alpha - 1)}, \quad (9)$$

with $*$ standing for complex conjugate. Due to the non-Hermitian nature of the operator, the first eigenmode is complex: $a_1 \equiv R_1 + iI_1$ with R_1 and I_1 being its real and imaginary parts, respectively. The remaining parameters in eq. (9) are $\beta \equiv R_1/I_1$ and the normalized frequency $\alpha \equiv \omega/I_1$.

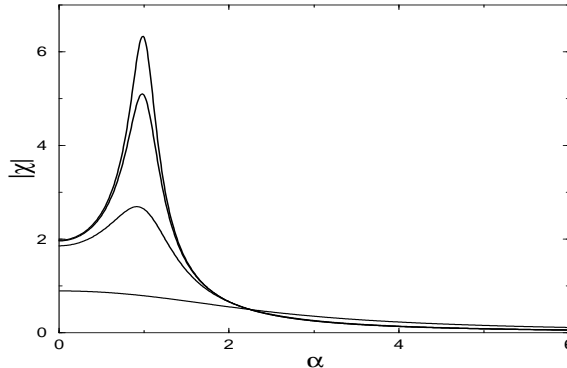


Fig. 1. Non-dimensional modulus of the susceptibility as a function of α , for different values of the parameter β , the smaller the value of β the sharper the curve. The resonance fades away practically for $\beta \approx 10$.

In Fig. 1, we show that during the relaxation process of non-equilibrium fluctuations the susceptibility undergoes a resonant behavior when the frequency of the force matches the imaginary part of the first eigenvalue of the non-perturbed operator \mathcal{L}_o . This behavior reveals the existence of a resonant coupling between the periodic force and the non-equilibrium source, responsible for the non-Hermitian nature of \mathcal{L}_o . The implications of that coupling in the energy transduction of the system will be analyzed in the next section.

3 Ratchet effect

Systems governed by the non-Hermitian dynamics discussed in the previous section may transduce the energy supplied by an unbiased periodic force into kinetic energy, thus inducing a net particle current. To analyze this peculiar behaviour, we will first calculate the power dissipated by the system. To that purpose we will apply the scheme of mesoscopic non-equilibrium thermodynamics [5]. For a system described by the probability density $\Psi(\mathbf{x}, t)$, the variation of the entropy due to changes in configurations in \mathbf{x} -space is given by

$$\delta S = -\frac{1}{T} \int \mu \delta \Psi d\mathbf{x}, \quad (10)$$

where $\mu = k_B T \ln \Psi + U$ is the chemical potential, $U(\mathbf{x}, t)$ the potential, and T the temperature. The rate of change of the entropy can be obtained by taking

the time derivative of eq. (10), and using eq. (1). One achieves

$$\frac{dS}{dt} + \int \psi \mathbf{v} \cdot \nabla(\mu/T) d\mathbf{x} = - \int \mathbf{J}_\psi \cdot \nabla(\mu/T) d\mathbf{x} , \quad (11)$$

The right hand side of eq. (11) constitutes the irreversible part of the rate of change of the entropy or entropy production. Consequently, the power supplied by the external force and dissipated into the system is obtained from eq. (11) by using the expression of the chemical potential. One obtains

$$P_F = \int \mathbf{J}_\psi \cdot \mathbf{F}(\mathbf{x}, t) d\mathbf{x} = \mathbf{F}_o \cdot \left\{ \frac{d}{dt} \langle \mathbf{x} \rangle - \langle \mathbf{v}(\mathbf{x}) \rangle \right\} \lambda(t) , \quad (12)$$

which defines the particle current $\langle \dot{\mathbf{x}} \rangle = d\langle \mathbf{x} \rangle / dt - \langle \mathbf{v}(\mathbf{x}) \rangle$. To obtain eq. (12), we have assumed an homogeneous force and used eqs. (1) and (2). Thus, P_F can be interpreted as the projection of the particle current along the direction of the oscillating force. The quantity of interest in experiments is the time-averaged dissipated power, defined as

$$P(\omega) = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} P_F dt , \quad (13)$$

This quantity is not only a function of the imaginary part of the susceptibility as occurs in Hermitian systems. The presence of the external drift introduces a more complicated dependence on the moments of \mathbf{x} . Moreover, it does not have a definite sign, and in general can be a non-monotonous function of the frequency expressing the resonant character of the energy dissipation. Since in general the external drift $\mathbf{v}(\mathbf{x})$ introduces a characteristic frequency in the system playing the same role of a vorticity, this systems behaves as a vortex ratchet.

4 Applications

Our purpose in this section is to present different manifestations of the vortex ratchet, as well as to indicate potential applications in different fields.

4.1 Orientable particles

An orientable particle of mesoscopic size in a vortex field under the influence of a periodic force exhibits the phenomenology discussed previously. In ab-

sence of the external force the orientation of the particle is characterized by a director vector undergoing Brownian motion which on average rotates at the velocity imposed by the local vorticity, $\boldsymbol{\omega}_o$. The most interesting situation occurs when the force is perpendicular to the vortex field. In that case, if $\omega > 1/2\omega_o$ the particle acquires an excess of angular velocity as a consequence of the torque it feels. In such a situation, the energy supplied by the periodic force is transduced into rotational kinetic energy, thus inducing a net particle current. This power is given by $P_F = \boldsymbol{\tau} \cdot (\boldsymbol{\Omega}_P - \frac{1}{2}\boldsymbol{\omega}_o)$, where $\boldsymbol{\Omega}_P$ is the average angular velocity of the particle, and $\boldsymbol{\tau}$ the torque acting on it. In this range of frequencies the system behaves as a Brownian motor.

The director could represent a dipole moment oriented by a field. Suspensions of such dipoles in a liquid phase exhibit peculiar collective behaviours. This is the case of electro- and magneto-rheological fluids, ferrofluids [6], [7], [8] and dilute solution of rod-like polymers [9].

4.2 *Itinerant oscillator models*

The itinerant oscillator model essentially consists of a Brownian particle with an orientable core coupled via an interaction potential to the shell. It was proposed to explain microwave dielectric absorption of polar fluids[10]. A particular realization of the model is an inhomogeneous body under the influence of a vortex field and a constant force perpendicular to it. The inhomogeneity induces a time-dependent dipole moment. The associated torque is $m(t)\mathbf{g} \times \mathbf{r}$, with $m(t)$ being the dipole moment strength, \mathbf{g} the constant external force, and \mathbf{r} the orientation vector. When the dipole moment varies periodically in time, as in the case of the orientable particle, the system behaves as a vortex ratchet. Unlike the previous case, variations in the exerted torque are due to internal reorganizations and not to variations of the external field. The itinerant oscillator model may mimic a living cell with an inhomogeneous density distribution in an external field, a particle in a cage formed by other particles, a rod-like polymer moving in a tube [9], or a monodomain magnetic particle whose magnetic moment undergoes fluctuations. Those systems would be good candidates to act as vortex ratchets.

4.3 *Diffusion through random and structured media*

A particle diffusing in a random media advected by a steady mean-flow velocity $\mathbf{v}_o(\mathbf{x})$ in the presence of a periodic force also manifests the ratchet effect previously discussed. The dynamics of the particle follows from the Fokker-Planck equation (3), where the drift $\mathbf{v}(\mathbf{x})$ is now a random velocity distributed around the mean-flow velocity according to a Gaussian probability distribution

with variance Γ . The Fourier transform of the random drift $\mathbf{v}(\mathbf{k})$ displays both longitudinal and transversal correlations

$$\langle v_i(\mathbf{k})v_j(\mathbf{k}') \rangle = 2(2\pi)^3 \Gamma \delta_{ij} \delta(\mathbf{k} + \mathbf{k}') . \quad (14)$$

The transversal correlations play the role of a vorticity, introducing the non-Hermiticity in the dynamics.

Assuming that the external force is homogeneous, the dissipation is obtained through eqs. (12) and (13), after averaging over the disorder. In Fig. 2, we have represented the normalized dissipated power P as a function of the normalized frequency α , for two levels of disorder. These levels are characterized by the parameter $s \equiv \sigma^2(Dk^2)^2$ where D is the diffusion coefficient and $\sigma^2 \equiv \Omega/2k^2\Gamma$, with Ω the volume of the system. The disorder then increases when decreasing s .

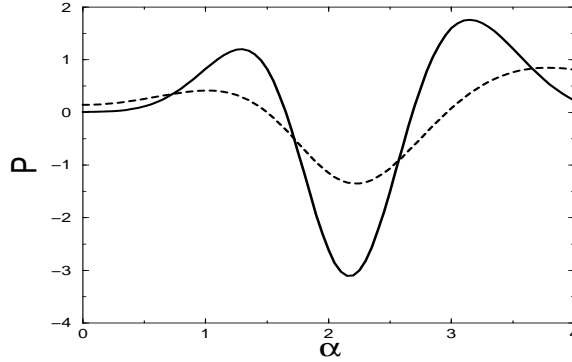


Fig. 2. Contribution of the external force to the dissipated energy as a function of α . Solid line corresponds to $s=2$, dashed line represents the same quantity for $s=1$.

The figure shows how the contribution of the periodic force to the total dissipated power exhibits a minimum when the frequency matches the characteristic frequency of the system, thus revealing the resonant character of the dissipation. Due to the characteristics of this system, when the power is negative the particle current is positive. The figure also shows that the power or, in view of eq. (12), the current is positive for frequencies around the resonance frequency, which manifests that the velocity of the particles is larger than the average drift. Therefore, in those conditions the system acts as a Brownian motor.

A similar phenomenon occurs in a spatially periodic two-dimensional pattern of triangular vortexes when a two-dimensional oscillating force is applied, at sufficiently high Reynolds number. In this conditions, a large-scale current appears. This effect is accompanied by a reduction of the dissipation in the system due to the induction of a negative eddy viscosity[11].

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